A New Approach to Functional Analysis on Graphs, the Connes-Spectral Triple and its Distance Function

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Abstract

We develop a certain piece of functional analysis on general graphs and use it to create what Connes calls a'*spectral triple*', i.e. a Hilbert space structure, a representation of a certain (function) algebra and a socalled '*Dirac operator*', encoding part of the geometric/algebraic properties of the graph. We derive in particular an explicit expression for the '*Connes-distance function*' and show that it is in general bounded from above by the ordinary distance on graphs (being, typically, strictly smaller(!) than the latter). We exhibit, among other things, the underlying reason for this phenomenon.

1 Introduction

In [1] we developed a version of discrete cellular network dynamics which is designed to mimic or implement certain aspects of Planck-scale physics (see also [2]). Our enterprise needed the developement of a variety of relatively advanced and not entirely common mathematical tools to cope with such complex and irregular structures as, say, *random graphs* and general networks.

On the mathematical side two central themes are the creation of an appropriate discrete analysis and a kind of discrete (differential) geometry or topology. The latter one comprises, among other things, e.g. a version of dimension theory for such objects (see [3]). It is the former one which turned out to be related to Connes' *noncommutative geometry* (see e.g. [4]) insofar as differential calculus on graphs (which are the underlying geometric structure of our physically motivated networks) is 'non-local', i.e. mildly non-commutative (for more details see [1]).

Graphs carry a natural *metric structure* given by a 'distance function' $d(x,y)$, x, y two *nodes* of the graph (see the following sections) and which we employed in e.g. [3] to develop dimensional concepts on graphs. Having Connes' concept of distance in noncommutative geometry in mind (cf. chapt. VI of [4]), it is a natural question to compute it in model systems, which means in our context: arbitrary graphs, and compare it with the already existing notion of graph distance mentioned above.

To this end one has, in a first step, to construct what Connes calls a 'spectral triple', in other words our first aim is it to recast part of the (functional) analysis on graphs (developed e.g. in [1] and in particular in section 3 of this paper as far as operator theory is concerned) in order to get both an interesting and natural Hilbert space structure , a corresponding representation of a certain (function) algebra and a natural candidate for a socalled 'Dirac operator' (not to be confused with the ordinary Dirac operator of the Dirac equation), which has to encode certain properties of the 'graph algebra'. This will be done in section 4.

In the last section, which deals with the distance concept deriving from the spectral triple, we will give this notion a closer inspection as far as graphs and similar spaces are concerned. In this connection some recent work should be mentioned, in which Connes' distance function was analyzed on certain simple models like e.g. one-dimensional lattices ([5]-[7]). These papers already show that it is a touchy business to isolate "the" correct Dirac operator and that it is perhaps worthwhile to scrutinize the whole topic in a more systematic way.

One of the advantages of our approach is that it both establishes a veritable piece of interesting functional analysis on graphs and yields a clear recipe how to calculate the 'Connes distance' in the most general cases ,exhibiting its true nature in this context as a non-trivial constraint on certain function classes on graphs. From this latter remark one can already conclude that its relation to the ordinary distance is far from trivial! We proved in fact that the Connes-distance is always bounded by the ordinary distance (being typically strictly smaller) and

clarified the underlying reason for this.

As to the mentioned papers [5] to [7] we would like to say that we realized their existence only after our manuscript has been completed, which should be apparent from the fact that our approach is sufficiently distinct. Therefore we prefer to develop our own line of reasonong in the paper under discussion and compare it in more detail with the other approaches elsewhere.

2 A brief Survey of Differential Calculus on Graphs

The following is a very sketchy compilation of certain concepts needed in the further analysis and may partly be new or known only to some experts in the field (even for specialists on graph theory the '*functional analysis-point of view*' is perhaps not entirely common). Our personal starting point is laid down in [1] and [2]. Two beautiful monographs about certain aspects of functional analysis on graphs (mostly on the level of finite matrix theory, i.e. typically being applied to finite graphs) are [8],[9]. As we are presently developing this fascinating field further into various directions in a more systematic way we refer the reader to forthcoming work as to more details.

2.1 Definition (Simple Graph): i) A *simple graph G* consists of a countable set of (*labelled*) *nodes* or *vertices* $\mathcal{V}$, elements denoted by e.g. n_i or $x, y \ldots$ and a set of *bonds* or *edges* $\mathcal{E}$ with E isomorphic to a subset of $\mathcal{V} \times \mathcal{V}$. We exclude the possibility of *elementary loops*, i.e. the above relation is *nonreflexive*. Stated differently, with x, y some nodes, $(x, y) \in \mathcal{E}$ implies $x \neq y$. Furthermore two nodes are connected by atmost one bond.
ii) In [1] we preferred to give the bonds an *orientation*, i.e. we can orient the bond connecting the nodes n_i, n_k by expressing it as b_{ik} or b_{ki} (this is not to be confused with the notion of *directed bonds*; see below). Concepts like these aquire their full relevance when the graph is given an *algebraic structure* ([1]).

2.2 Algebraic Orientation: In a sense to be specified below we postulate

$$b_{ik} = -b_{ki} \tag{1}$$

2.3 First Step of a Differential Structure: i) We can now define the *Node(Vertex)-Space* C_0 and the *Bond(Edge)-Space* C_1 by considering functions from $\mathcal{V}$ and $\mathcal{E}$ to certain spaces, in the most simplest case the complex numbers $\mathbb{C}$. (Note that many other choices are possible; the above is only the choice we make for convenience in the following!). As our basic sets are discrete, the elementary building blocks are the *indicator functions* n_i, b_{ik} itself (see [1]), i.e. which have the value 1 on n_i or b_{ik} and being zero elsewhere.

Consequence: C_0, C_1 can be regarded as (complex) vector spaces with basis elements n_i, b_{ik}.

ii) We introduce two operators d, δ, well-known from algebraic topology (see [1] for more motivation). In a first step they are given on the basis elements and then extended linearly:

$$\delta : \ b_{ik} \ \rightarrow \ n_k - n_i \tag{2}$$

$$d : \ n_i \ \rightarrow \ \sum_k b_{ki} \tag{3}$$

the sum running over the nodes n_k being directly connected with n_i by a bond. We see that δ maps C_1 into C_0 while d maps C_0 into C_1.

2.4 Observation: That d has in fact the character of a differential operation can be seen by the following identity ([1]):

$$df = 1/2 \cdot \sum_{ik} (f_k - f_i) b_{ik} \tag{4}$$

where the factor $1/2$ arises only from the symmetric summation over i and k which counts (for convenience) each bond twice on the rhs and where the above relation $b_{ik} = -b_{ki}$ has been employed.

As long as we remain on the level of pure vector spaces the above choice is probably the most natural one. But if we try to make the framework into a full differential calculus (e.g. in the spirit of noncommutative geometry) we need something like a (left-,right-)module structure, i.e. we have to multiply elements from C_1 from the left/right with functions from C_0 in order to arrive at something what is called a *differential algebra* (with e.g. a *Leibniz rule* to hold). These things have been discussed in more detail in [1]; for a different approach see also [10], which is more in the spirit of Connes ([4]).

To this end we have to extend or rather embed the space C_1 (in)to a larger C_1' with basis elements denoted by us as d_{ik}.

2.5 Definition/Observation: For various reasons ([1]) we have to enlarge the space C_1 to C_1' with new basis elements d_{ik}, but now with d_{ik} being linearly independent from d_{ki} and

$$b_{ik} := d_{ik} - d_{ki} \tag{5}$$

a) Pictorially d_{ik}, d_{ki} may be considered as *directed bonds* (in contrast to orientable bonds) having the fixed(!) direction from, say, n_i to n_k and vice versa.

b) Considering the whole context rather from the viewpoint of (discrete) manifolds, we defined the d_{ik} as linear forms, mapping the *tangential basis vectors* $\partial_{ik'}$ attached to the node n_i onto $\delta_{kk'}$, i.e:

$$< d_{ki} | \partial_{ik'} > = \delta_{kk'} \tag{6}$$

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In this sense the d_{ik}, d_{ki} can be regarded as objects being attached to the nodes n_i, n_k respectively, while the bonds b_{ik} are delocalized, living in the "environment" between the nodes. We think that this property is of physical significance and the crucial difference between d_{ik} and b_{ik}.

In this new basis the *differential df* of Observation 2.4 acquires the form

2.6 Corollary:

$$df = \sum_{ik}(f_k - f_i)d_{ik} \tag{7}$$

2.7 Lemma: It is obvious that the map δ can be canonically extended to this larger space by:

$$\delta_1 : \ d_{ik} \rightarrow \ n_k \tag{8}$$

A last but important point we want to mention is Observation 3.13 of [1]:

2.8 Observation(Graph-Laplacian):

$$\delta df = -\sum_i(\sum_k f_k - v_i \cdot f_i)n_i = -\sum_i(\sum_k(f_k - f_i))n_i =: -\Delta f \tag{9}$$

where v_i denotes the *node(vertex) degree* or the *valency* of the node n_i, i.e. the number of nearest neighbors n_k being connected to it by a bond.

It is interesting that this graph laplacian, which we developed following a completely different line of reasoning in [1], is intimately connected with an object wellknown to graph theorists, i.e. the *adjacency matrix* of a graph.

2.9 Definition(Adjacency Matrix): i)The entries a_{ik} of the *adjacency matrix* A have the value one if the nodes n_i, n_k are connected by a bond, zero elsewhere. If the graph is *undirected* (but orientable; the case we mainly discuss), the relation between n_i, n_k is *symmetric*, i.e.

$$a_{ik} = 1 \quad \Rightarrow \quad a_{ki} = 1 \quad \text{etc.} \tag{10}$$

with the consequence:
ii) If the graph is *simple* and *undirected*, A is a symmetric matrix with zero diagonal elements.

Remark: More general A's occur if more general graphs are discussed.

Observation: With our definition of Δ it holds:

$$\Delta = A - V \tag{11}$$

where V is the diagonal *degree matrix*, having v_i as diagonal entries.

Proof: As we have not yet introduced a Hilbert space structure (which we will

do below), the proof has to be understood, for the time being, in an algebraic way. We then have:

$$Af = A(\sum f_i n_i) = \sum_i (\sum_{k-i} n_k) = \sum_i (\sum_{k-i} f_k) n_i \tag{12}$$

$$Vf = \sum_i (v_i f_i) n_i \tag{13}$$

hence the result.

Remarks:i) Here and in the following we use the abbreviation $k - i$ if the nodes n_k, n_i are connected by a bond, the summation always extending over the first variable.

ii) From this interplay between *graph geometry* and *functional analysis* follow a lot of deep and fascinating results, as is always the case in mathematics if two seemingly well separated fields turn out to be closely linked on a deeper level. This is particularly the case if geometry is linked with algebra or functional analysis.

3   Some Functional Analysis on Graphs

After the preliminary remarks made in the previous section we now enter the heart of the matter. Our first task consists of endowing a general graph with both a sufficiently reach and natural Hilbert space structure on which the various operators to be constructed in the following can act.

3.1 Definition (Hilbert Space): i)In C_0 we choose the subspace H_0 of sequences f so that:

$$\|f\|^2 = \sum |f_i|^2 < \infty \tag{14}$$

ii) In C_1, C_1' respectively we make the analogous choice:

$$H_1 := \{g| \, \|g\|^2 := \sum |g_{ik}|^2 < \infty; \, g_{ik} = -g_{ki}\} \tag{15}$$

$$H_1' := \{g'| \, \|g'\|^2 := \sum |g_{ik}'|^2 < \infty\} \tag{16}$$

with $g = \sum g_{ik} d_{ik}$, $g' = \sum g_{ik}' d_{ik}$ and the respective ON-bases $\{n_i\}$, $\{d_{ik}\}$, that is $< d_{ik}|d_{i'k'} >= \delta_{ii'}\delta_{kk'}$.

Remark: The convention in ii) is made for convenience in order to comply with our assumption $b_{ik} = -b_{ki}$, which is to reflect that the bonds b are undirected and that functions over it should be given modulo their possible orientation! Members of H_1 can hence also be written

$$\sum g_{ik} d_{ik} = 1/2 \sum g_{ik} d_{ik} + 1/2 \sum g_{ki} d_{ki} = 1/2 \sum g_{ik}(d_{ik} - d_{ki}) = 1/2 \sum g_{ik} b_{ik} \tag{17}$$

iii) As H, H' we take the *direct sums*:

$$H := H_0 \oplus H_1, \ H' := H_0 \oplus H_1' \tag{18}$$

3.2 Observation: Obviously H_1 is a subspace of H_1' and we have

$$< b_{ik} | b_{ik} >= 2 \tag{19}$$

i.e. the b_{ik} are not(!) normalized if the d_{ik} are. We could of course enforce this but then a factor two would enter elsewhere.

With these definitions it is now possible to regard the maps d, δ as full-fledged operators between these Hilbert (sub)spaces.

3.3 Assumption: To avoid domain problems and as it is natural anyhow, we assume from now on that the *node degree* $v(n_i)$ is *uniformly bounded* on the graph G, i.e.

$$v_i < v_{max} \text{ for all } i \tag{20}$$

3.4 Definition/Observation: i)

$$d: \ H_0 \to H_1, \quad \delta: \ H_1 \to H_0 \tag{21}$$

ii) $d_{1,2}$ with

$$d_{1,2}: \ n_i \to \sum d_{ki}, \ \sum d_{ik} \tag{22}$$

respectively and linearly extended, are operators from $H_0 \to H_1'$ and we have

$$d = d_1 - d_2 \tag{23}$$

iii) δ may be extended in a similar way to H_1' via:

$$\delta_{1,2}: \ d_{ik} \to n_k, \ n_i \tag{24}$$

respectively and linearly extended. We then have:

$$\delta_1(b_{ik}) = \delta(b_{ik}) = n_k - n_i = (\delta_1 - \delta_2)(d_{ik}) \tag{25}$$

It is remarkable that $v_i \leq v_{max}$ implies that all the above operators are *bounded*(!) (in contrast to similar operators in the continuum, which are typically unbounded).

3.5 Theorem: All the operators introduced above are bounded on the respective Hilbert spaces, i.e. their domains are the full Hilbert spaces under discussion.

Proof: We prove this for, say, d; the other proofs are more or less equivalent.

$$d: \ H_0 \ni \sum f_i n_i \to \sum_i (\sum_{k=1}^{v_i} b_{ki}) = \sum_{ik} (f_k - f_i) d_{ik} \tag{26}$$

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and for the norm of the rhs:

$$\|rhs\|^2 = \sum_{ik} |(f_k - f_i)|^2 = \sum_{ik}(|f_i|^2 + |f_k|^2 - \overline{f_k}f_i - \overline{f_i}f_k) = 2 \cdot \sum_i v_i |f_i|^2 - 2 \cdot \sum_{ik} \overline{f_k}f_i \tag{27}$$

The last expression can be written as:

$$\|df\|^2 = 2(<f|Vf> - <f|Af>) = <f| -2\Delta f> \tag{28}$$

which is a remarkable result. It shows, among other things, that d and its norm are closely connected with the *expectation values* of the *adjacency* and *degree matrix* respectively the *graph Laplacian* (introduced in the previous section). This is, of course, no accident and relations like these will be clarified more systematically immediately.

It follows already from the above that we have:

3.6 Observation:

$$\|df\|^2 = <f|d^*df> = <f| -2\Delta f> \tag{29}$$

i.e.

$$d^*d = -2\Delta \text{ and } \|d\|^2 = \sup_{\|f\|=1} <f| -2\Delta f> = \| -2\Delta\| \tag{30}$$

We then have:

$$0 < \sup_{\|f\|=1} <f| -2\Delta f> \leq 2v_{max} + 2 \sup_{\|f\|=1} | <f|Af> | \tag{31}$$

Due to our assumption A is a (in general infinite) hermitean matrix with entries 0 or 1, but with atmost v_{max} nonzero entries in each row and vanishing diagonal elements.

Remark: i)It is possible to treat also more general A's if we admit more general graphs (e.g. socalled *multi graphs*; see [8] and [9]).
ii) It should be noted that, whereas A is a matrix with nonnegative entries, it is not(!) *positive* in the sense of *linear operators*. The positive(!) operator in our context is the Laplacian $-\Delta$, which can be seen from the above representation as $1/2d^*d$. On the other side matrices like A are frequently called positive in the matrix literature, which is, in our view, rather missleading.
iii) It should further be noted that we exclusively use the *operator norm* for matrices (in contrast to most of the matrix literature), which may also be called the *spectral norm*. It is unique insofar as it coincides with the socalled *spectral radius* (cf. e.g. [11] or [12]), that is

$$\|A\| := \sup\{|\lambda|; \lambda \in spectr(A)\} \tag{32}$$

After these preliminary remarks we will now estimate the norm of A. To this end we exploit a simple but effective inequality, which perhaps better known in numerical mathematics and which we adapt to infinite matrices of the above type.

3.7 Theorem (Variant of Gerschgorin Inequality): Let A be a finite adjacency matrix with admost v_{max} nonzero entries in each row. Then:

$$\sup\{|\lambda|; \lambda \text{ an eigenvalue}\} \leq v_{max} \tag{33}$$

Proof: This is an immediate application of the original Gerschgorin inequality to our case (vanishing diagonal elements; see e.g. [13]).

To treat the infinite case with finite v_{max}, we choose a sequence of n-dimensional subspaces X_n with basis elements $e_1, \ldots e_n$. In these subspaces the corresponding projections A_n of the infinite A fulfills the above assumption. We then have for a normalizable vector x; $\sum |x_i|^2 < \infty$ that

$$< x|A_n x > \rightarrow < x|Ax > \tag{34}$$

The same does then hold for functions of A, A_n, in particular for the spectral projections; hence:

3.8 Theorem (Norm of A): With the adjacency matrix A possibly infinite and a finite v_{max} we have the important estimate:

$$\|A\| = \sup\{|\lambda|; \lambda \in \ spectr(A)\} \leq v_{max} \tag{35}$$

This result concludes also the proof of theorem 3.5!

The above reasoning shows that many of the technical difficulties (e.g. domain problems) are absent on graphs with uniformly bounded degree as all the operators turn out to be bounded under this premise. We want to conclude this section with deriving some relations among the operators introduced in Definition/Observation 3.4:

We already realized that

$$d^* d = -2\Lambda \tag{36}$$

holds, but we did not make explicit the true nature of d^*. On the one side

$$d : H_0 \rightarrow H_1 \ , \ d^* : H_1 \rightarrow H_0 \tag{37}$$

On the other side

$$\delta : H_1 \rightarrow H_0 \tag{38}$$

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and for $g \in H_1$, i.e. $g_{ik} = -g_{ki}$:

$$\sum_{ik} g_{ik} d_{ik} = 1/2 \sum_{ik} g_{ik}(d_{ik} - d_{ki}) = 1/2 \sum_{ik} g_{ik} b_{ik} \tag{39}$$

Calculating now

$$< g | df > = \sum g_{ik}(f_k - f_i) = < 2\delta g | f > \tag{40}$$

we can infer:

3.9 Observation: i)The adjoint d^* of d with respect to the spaces H_0, H_1 is 2δ, which also follows from the comparison of the representation of the graph Laplacian in Observation 2.8 and Definition/Observation 3.4, i.e:

$$\delta d = -\Delta = 1/2 d^* d \tag{41}$$

ii) For the natural extensions $d_{1,2}, \delta_{1,2}$ of d, δ to the larger (sub)spaces H_0, H_1' we have (cf. the definitions in Definition/Observation 3.4):

$$\delta_1 = (d_1)^* \ , \ \delta_2 = (d_2)^* \tag{42}$$

hence

$$(\delta_1 - \delta_2) = (d_1 - d_2)^* = d^* \tag{43}$$

Proof of ii):

$$< g | d_1 f > = \sum_{ik} g_{ik} f_i = < \delta_1 g | f > \quad \text{etc.} \tag{44}$$

In other words, the important result for the following is that with respect to the larger space H_1' the adjoint d^* of d is $(\delta_1 - \delta_2)$ and we have:

$$(\delta_1 - \delta_2) b_{ik} = 2(n_k - n_i) = 2\delta b_{ik} \tag{45}$$

Henceforth we identify d^* with $(\delta_1 - \delta_2)$.

4   The Spectral Triple on a general Graph

The Hilbert space under discussion in the following is

$$H = H_0 \oplus H_1' \tag{46}$$

The *natural representation* of the function algebra $\mathcal{F}$

$$\{f; f \in \mathcal{C}_0, \sup_i |f_i| < \infty\} \tag{47}$$

on H is given by:

$$H_0 : \ f \cdot f' = \sum f_i f_i' \cdot n_i \text{ for } f' \in H_0 \tag{48}$$

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$$H_1' : \ f \cdot \sum g_{ik}d_{ik} := \sum f_i g_{ik}d_{ik} \tag{49}$$

From previous work ([1]) we know that $\mathcal{C}_1'$ carries also a right-module structure, given by:

$$\sum g_{ik}d_{ik} \cdot f := \sum g_{ik}f_k \cdot d_{ik} \tag{50}$$

Remark: For convenience we do not distinguish notationally between elements of $\mathcal{F}$ and their Hilbert space representations.

The important and nontrivial object is the socalled *Dirac operator D*. As D we will take the operator:

$$D := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \tag{51}$$

acting on

$$H = \begin{pmatrix} H_0 \\ H_1' \end{pmatrix} \tag{52}$$

with

$$d^* = (\delta_1 - \delta_2) \tag{53}$$

4.1 Definition/Observation (Spectral Triple): Our *spectral triple* on a general graph is given by

$$(H, \mathcal{F}, \mathcal{D}) \tag{54}$$

introduced in the preceeding formulas.

As can be seen from the above, the connection with the graph Laplacian is relatively close since:

$$D^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \tag{55}$$

and

$$d^*d = -2\Delta \tag{56}$$

dd^* is the corresponding object on H_1'.

We now calculate the commutator $[D, f]$ on an element $f' \in H_0$:

$$(d \cdot f)f' = \sum_{ik}(f_k f_k' - f_i f_i')d_{ik} \tag{57}$$

$$(f \cdot d)f' = \sum_{ik} f_i(f_k' - f_i')d_{ik} \tag{58}$$

hence

$$[D, f]f' = \sum_{ik}(f_k f_k' - f_i f_k')d_{ik} \tag{59}$$

On the other side the right-module structure yields:

$$df \cdot f' = \left(\sum_{ik}(f_k - f_i)d_{ik}\right) \cdot \left(\sum_k f_k' n_k\right) = \sum_{ik}(f_k f_k' - f_i f_k')d_{ik} \tag{60}$$

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In a next step one has to define df as operator on H'_1. This is done in the following way (the reason follows below):

$$df : \ d_{ik} \rightarrow (f_i - f_k)n_k \tag{61}$$

and linear extension.

4.2 Definition: The representation of df on H is defined in the following way:

$$df : \ H_0 \rightarrow H'_1 \ , \ H'_1 \rightarrow H_0 \quad \text{via} \tag{62}$$

$$n_k \rightarrow (f_k - f_i)d_{ik} \ , \ d_{ik} \rightarrow (f_i - f_k)n_k \tag{63}$$

We are now able to calculate the commutator $[D, f]$ on H'_1:

$$(d^* \cdot f)g - (f \cdot d^*)g = \sum (f_i - f_k)g_{ik} \cdot n_k \tag{64}$$

with $g = \sum g_{ik}d_{ik} \in H'_1$.
On the other side:

$$(df)g = \sum g_{ik}(df)d_{ik} = \sum (f_i - f_k)g_{ik} \cdot n_k \tag{65}$$

We hence have the important result:

4.3 Theorem (Dirac Operator): With the definition of the representation of df on H as in Definition 4.2 it holds:

$$[D, f]x = (df)x \ , \ x \quad \text{being an element of} \quad H \tag{66}$$

and in particular:

$$[d, f] = df|_{H_0} \ , \ [d^*, f] = df|_{H'_1} \tag{67}$$

with the two maps intertwining H_0 and H'_1.

Remark: In the above sections we have discussed a Hilbert space representation based on the Hilbert space $H_0 \oplus H'_1$ and operators d, d^* respectively df etc. In the same way one could choose a '*dual representation*' over the '*tangential space*' built from the ∂_{ik}'s (introduced in [1]; see also formula (6) in Definition/Observation 2.5). In this case the operator d goes over into the dual object ∇. This choice is natural as well and arises from a '*dualization*' of the above one. It will be discussed elsewhere in connection with the developement of a discrete Euler-Lagrange formalism on graphs.

5 The Connes-Distance Function on Graphs

The first step consists of calculating the norm of $[D, f] = df$ respectively its supremum under the condition $\|f\| \leq 1$. We have (with the help of previous calculations):

$$[D, f] = \begin{pmatrix} 0 & [d^*, f] \\ [d, f] & 0 \end{pmatrix} \tag{68}$$

and with $\{f_i\}$ uniformly bounded and real, i.e. f a bounded and s.a. operator:

$$[d, f]^* = -[d^*, f] \tag{69}$$

From the general theory we have:

$$\|A\| = \|A^*\| \tag{70}$$

(where here and in the following A denotes a general operator and not(!) the adjacency matrix) hence

5.1 Observation:
$$\|[d, f]\| = \|[d^*, f]\| \tag{71}$$

and

$$\|[D, f](X)\|^2 = \|[d, f]x\|^2 + \|[d^*, f]y\|^2 \tag{72}$$

with

$$X := \begin{pmatrix} x \\ y \end{pmatrix} \tag{73}$$

Choosing the abbreviation $A := [d, f]$, the norm of $[D, f]$ is:

$$\|[D, f]\|^2 = \sup\{\|Ax\|^2 + \|A^*y\|^2; \|x\|^2 + \|y\|^2 = 1\} \tag{74}$$

Normalizing x, y to $\|x\| = \|y\| = 1$ and representing a general normalized vector X as:

$$X = \lambda x + \mu y , \ \lambda, \mu > 0 \text{ and } \lambda^2 + \mu^2 = 1 \tag{75}$$

we get:

$$\|[D, f]\|^2 = \sup\{\lambda^2 \|Ax\|^2 + \mu^2 \|A^*y\|^2; \|x\| = \|y\| = 1, \lambda^2 + \mu^2 = 1\} \tag{76}$$

where now x, y can be varied independently of λ, μ in their respective admissible sets, hence:

5.2 Conclusion:

$$\|[D, f]\|^2 = \|A\|^2 = \|[d, f]\|^2 = \|[d^*, f]\|^2 \tag{77}$$

where the operators on the rhs act in the reduced spaces H_0, H_1' respectively.

It follows that in calculating $\|[D, f]\|$ one can restrict oneself to the easier to handle $\|[d, f]\|$. For the latter expression we then get from the above ($x \in H_0$):

$$\|df \cdot x\|^2 = \sum_i (\sum_{k=1}^{v_i} (f_i - f_k)^2) |x_i|^2 \tag{78}$$

Abbreviating

$$\sum_{k=1}^{v_i} (f_k - f_i)^2 =: a_i \geq 0 \tag{79}$$

and calling the supremum over i a_s, it follows:

$$\|df \cdot x\|^2 = a_s \cdot (\sum_i a_i/a_s \cdot |x_i|^2) \leq a_s \tag{80}$$

for $\|x\|^2 = \sum_i |x_i|^2 = 1$.

On the other side, choosing a sequence of normalized basis vectors x_ν so that the corresponding a_ν converges to a_s we get:

$$\|df \cdot x_\nu\|^2 \to a_s \tag{81}$$

5.3 Theorem (Norm of $\|[D, f]\|$):

$$\|[D, f]\| = \sup_i (\sum_{k=1}^{v_i} (f_k - f_i)^2)^{1/2} \tag{82}$$

The *'Connes-distance function'* is now defined as follows:

5.4 Definition (Connes-distance function):

$$dist_C(n, n') := \sup\{|f_{n'} - f_n|; \|[D, f]\| = \|df\| \leq 1\} \tag{83}$$

where n, n' are two arbitrary nodes on the graph.

Remark: It is easy to prove that this defines a metric on the graph.

5.5 Corollary: It is sufficient to vary only over the set $\{f; \|f\| = 1\}$.

Proof: This follows from

$$|f_k - f_i| = c \cdot |f_k/c - f_i/c| \; ; \; c = \|df\| \tag{84}$$

and

$$\|d(f/c)\| = c^{-1}\|df\| = 1 \tag{85}$$

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with $c \leq 1$ in our case.

It is evidently a nontrivial task to calculate this distance on an arbitrary graph as the above constraint is quite subtle whereas it is given in a closed form. We refrain at this place from a complete discussion but add only the following remarks concerning the connection to the ordinary distance function introduced in the beginning of the paper.

Having an admissible function f so that $\sup_i(\sum_{k=1}^{v_i}(f_k - f_i)^2)^{1/2} \leq 1$, this implies that, taking a 'minimal path' γ from, say, n to n', the jumps $|f_{\nu+1} - f_\nu|$ between neighboring nodes along the path have to fulfill:

$$|f_{\nu+1} - f_\nu| \leq 1 \tag{86}$$

and are typically strictly smaller than 1.

On the other side the Connes distance would only become identical to the ordinary distance $d(n, n')$ if there exist a sequence of admissible node functions with these jumps approaching the value 1 along such a path. Only in this case one would get:

$$\sum_\gamma |f_{\nu+1} - f_\nu| \to \sum_\gamma 1 = length(\gamma) \tag{87}$$

The construction of such functions is however an intricate and "nonlocal" business if at the same time the above constraint concerning the jumps between neighboring nodes is to be fulfilled.

5.6 Observation (Connes-distance): In general one has the inequality

$$dist_C(n, n') \leq d(n, n') \tag{88}$$

This result can e.g. be tested in a simple example. Take, say, a square with vertices and edges:

$$x_1 - x_2 - x_3 - x_4 - x_1 \tag{89}$$

Let us calculate the Connes-distance between x_1 and x_3.

As the sup is taken over functions(!) the summation over elementary jumps is (or rather: has to be) pathindependent (this is in fact both a subtle and crucial constraint for practical calculations). It is an easy exercise to see that the *sup* can be found in the class where the two paths between x_1, x_3 have the 'valuations' $(1 \geq a \geq 0)$:

$$x_1 - x_2 : a \ , \ x_2 - x_3 : (1 - a^2)^{1/2} \tag{90}$$

$$x_1 - x_4 : (1 - a^2)^{1/2} \ , \ x_4 - x_3 : a \tag{91}$$

Hence one has to find $\sup_{0 \leq a \leq 1}(a + \sqrt{1 - a^2})$. Setting the derivative with respect to a to zero one gets $a = \sqrt{1/2}$. Hence:

5.7 Example (Connes-distance on a square):

$$dist_C(x_1, x_3) = \sqrt{2} < 2 = d(x_1, x_3) \tag{92}$$

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